Example 9: More Than Meets the Eye

Task

In earlier grades you may have seen problems that asked you to find the next term in a sequence, such as 3, 7, 11. One possible answer is 15, assuming the sequence is generated by evaluating $f(x) = 4x - 1$ at $x = 1, 2, 3$. But are other answers possible?

In the Classroom (Fourth-Year Mathematics)

Teacher: Find the sequence generated by evaluating $g(x) = x^3 - 6x^2 + 15x - 7$ at $x = 1, 2, 3$.

Student: I get $g(1) = 3$, $g(2) = 7$, and $g(3) = 11$. It's the same sequence: 3, 7, 11.

Teacher: What would be the next term if you used $g(x)$ instead of $f(x)$?

Student: It would be $g(4) = 21$. That's different from what we get by using $f(x)$.

Teacher: Can you find other polynomials that generate the sequence 3, 7, 11?

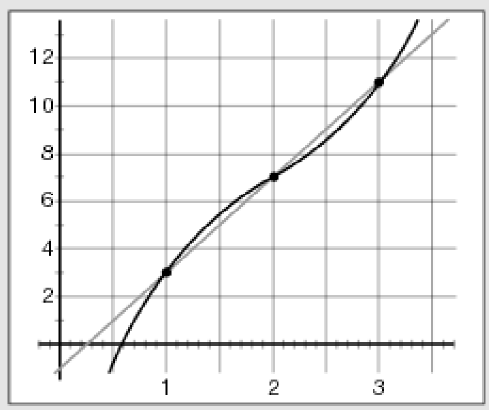
Student: I don't see how you came up with that weird cubic for g in the first place.

Teacher: What does it mean when we say that $g(1) = 3$?

Student: Well, it means that the y-value, or output, is 3 when the x-value, or input, is 1.

Teacher: So how can two different functions, f and g, have the same values at $x = 1, 2,$ and 3?

Student: I suppose that both of their graphs would have to have the same y-values…. Hey, that means they must intersect at those three points! Let me check that by graphing them. Yes, when I graph them, I see that the straight line graph of $f(x)$ intersects the cubic graph of $g(x)$ at $x = 1, 2,$ and 3.



Teacher: Can you see now how you might find another polynomial graphically?

Student: Maybe I could figure out a way to change the shape of the cubic graph but keep the intersection points the same.

Teacher: How would you do that algebraically?

Student: I could try to stretch the difference between $f(x)$ and $g(x)$, which is $g(x) - f(x) = x^3 - 6x^2 + 11x - 6$. I could triple that, for example, and add it back on to $f(x)$ and get $4x - 1 + 3(x^3 - 6x^2 + 11x - 6) = 3x^3 - 18x^2 + 37x - 19$.

Example 9: More Than Meets the Eye—*Continued*

Teacher: Excellent. Now I want to understand what is really going on here. Is there anything special about the polynomial $x^3 - 6x^2 + 11x - 6$ that makes this work?

Student: Not that I can see.

Teacher: Why don't you try factoring it on your CAS?

▪ f(x)=4x-1	f(x)=4·x-1
▪ g(x)=x^3-6x^2+15x-7	$g(x)=x^3-6\cdot x^2+15\cdot x-7$
▪ h(x)=g(x)-f(x)	$h(x)=x^3-6\cdot x^2+11\cdot x-6$
▪ factor(h(x))	$(x-3)\cdot(x-2)\cdot(x-1)$
▪ j(x)=f(x)+3*h(x)	$j(x)=3\cdot x^3-18\cdot x^2+37\cdot x-19$
▪ j(1)	3
▪ j(2)	7
▪ j(3)	11
▪	

Student: I get $x^3 - 6x^2 + 11x - 6 = (x - 1)(x - 2)(x - 3)$. Oh, I see. When I look at the factored form, I can see that the difference between $f(x)$ and $g(x)$ is 0 at x = 1, 2, and 3. So I could get many polynomials that generate the same sequence just by adding a multiple of this polynomial to $f(x)$.

Teacher: What is the general form of such a polynomial?

Student: It is $4x - 1 + k(x - 1) \cdot (x - 2)(x - 3)$, where k can be any real number. When I look at my graphing program, I notice that as k increases, the graph of the polynomial stretches away from the line.